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LOWER SEMICONTINUOUS PERTURBATIONS OF MAXIMAL MONOTONE DIFFERENTIAL INCLUSIONS

BY

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ABSTRACT

We prove existence of solutions to

$$\dot{x} \in -Ax + F(t,x),$$
$$x(a) = x^0,$$

where A is a maximal monotone operator in $\mathbf{R}^n$ and F is a multifunction measurable in (t,x) and l.s.c. in x, satisfying a sublinear growth condition.

1. Introduction

In [6], Cellina and Marchi proved an existence result for differential inclusions of the form

$$\dot{x} \in -Ax + F(t,x), \tag{1}$$

where A is a maximal monotone operator and F is a continuous map with compact (not necessarily convex) values which verifies a sublinear growth condition. The main tool used in their proof is a continuous selection theorem for the map

$$x \mapsto \{u \in L^1(I) : u(t) \in F(t, x(t)) \text{ a.e.}\} \tag{2}$$

defined on a compact subset of $L^1(I, \mathbf{R}^n)$. This approach goes back to a paper of Antosiewicz and Cellina [1], who considered the special case $A = 0$ with no

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convexity assumptions on the values of F. The results in [1] were generalized by Bressan [3] and Lojasiewicz [9] assuming the map F to be:

(a) jointly measurable in (t, x),

(b) lower semicontinuous in x.

In this paper we show that (1) still has a solution if A is a maximal monotone operator and F satisfies only (a) and (b) above and the same sublinear growth condition. Our proof follows the same fixed point argument of [6] and is based on a selection theorem of Fryszkowski [7], which contains the selection theorems used in [1], [3] and [6]. In fact, Fryszkowski's result permits a general treatment of operators of the type (2). The main part of this paper consists thus in proving that the operator (2) satisfies the assumptions of Fryszkowski's theorem.

2. Assumptions and statement of the main result

In what follows, A is a maximal monotone operator in $\mathbf{R}^n$, i.e. a set-valued map from a subset $D(A)$ of $\mathbf{R}^n$ into the subsets of $\mathbf{R}^n$, with the following two properties:

(A1) $\forall x_1,\ \ x_2 \in D(A),\ \ \ \forall v_i \in Ax_i,\ \ i = 1, 2,$

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geqq 0;$$

(A2) the range of $I + A$ is all of $\mathbf{R}^n$.

It is known that $\overline{D(A)}$ is convex, and that Ax is convex closed for any $x \in D(A)$ (see [2]).

We will consider a map F from $[a, +\infty) \times \overline{D(A)}$ into the compact subsets of $\mathbf{R}^n$ with the following properties:

(F1) $F(.,.)$ is $\mathscr{L} \otimes \mathscr{B}$-measurable, i.e. for any closed set $C \subset \mathbf{R}^n$ the set

$$F^-(C) := \{(t, x) \in [a, +\infty) \times \overline{D(A)} : F(t, x) \cap C \neq \varnothing\}$$

belongs to the σ-algebra generated by the sets of the form $L \times B$, where L is a Lebesgue measurable subset of $[a, +\infty)$ and B is a Borel subset of $\overline{D(A)}$;

(F2) for each $t \geqq a$, $F(t, .)$ is lower semicontinuous, i.e. for any closed set $C \subset \mathbf{R}^n$ the set

$$F(t, .)^+(C) := \{x \in \overline{D(A)} : F(t, x) \subset C\}$$

is closed in $\mathbf{R}^n$;

(F3) there exist two non-negative locally integrable functions α, β: $[a, +\infty) \to \mathbf{R}$ such that, for every $(t, x) \in [a, +\infty) \times \overline{D(A)}$,

$$|F(t, x)| := \sup\{|y| : y \in F(t, x)\} \leqq \alpha(t)|x| + \beta(t).$$

In the present paper we study the existence of solutions to the initial value problem

$$\text{(P)} \qquad \dot{x} \in -Ax + F(t, x), \qquad x(a) = x^0 \in \overline{D(A)}.$$

By a solution of (P) we mean a function $x \in C([a, +\infty), \mathbf{R}^n)$ which is absolutely continuous on every compact subset of $(a, +\infty)$ and is such that $x(a) = x^0$ and $x(t) \in D(A)$ for a.e. $t > a$ and, for some measurable selection $f(\,.\,)$ from $F(\,.\,, x(\,.\,))$,

$$(\mathrm{P}_f) \qquad \dot{x} \in -Ax + f(t) \qquad \text{for a.e. } t \geqq a$$

(see [2] and [6]).

Our main result is the following:

THEOREM 2.1. *If A is a maximal monotone operator and* (F1)–(F3) *hold, then problem* (P) *has a solution for any $x^0 \in \overline{D(A)}$.*

3. Some known results

In this section we state some known facts which will be used in the following. The first lemma illustrates the properties of a maximal monotone differential inclusion. For any compact interval I in $[a, +\infty)$, we denote by $|\cdot|_{i,I}$ the usual norm in $L^i(I) := L^i(I, \mathbf{R}^n)$, and we set $L^i_{\mathrm{loc}}([a, +\infty)) := L^i_{\mathrm{loc}}([a, +\infty), \mathbf{R}^n)$ ($i = 1$ or $i = \infty$).

LEMMA 3.1 ([2, Thm 1.2]). *For any $f \in L^1_{\mathrm{loc}}([a, +\infty))$ and any initial value $x^0 \in \overline{D(A)}$ there exists a unique solution u_f to* (P_f). *For every $t \geqq a$,*

$$|u_f(t) - u_g(t)| \leqq \int_a^t |f(s) - g(s)|\, ds$$

and, given any interval $I := [\tau, \tau + T]$, there exists a constant C depending only on A such that

$$|\dot{u}_f|_{1,I} \leqq C[(1 + T + |f|_{1,I}) \cdot (1 + |u_f|_{\infty,I}) + |u_f(\tau)|^2].$$

As a straightforward consequence of Lemma 3.1 we have that the map

$$i : L^1_{loc}([a, +\infty)) \to L^1_{loc}([a, +\infty))$$
$$f \mapsto u_f$$

is well-defined and continuous. The next lemma gives a kind of *a priori* estimate on the solutions of (P_f). We denote by $u_0(\cdot)$ the solution of (P_f) with $f = 0$.

LEMMA 3.2 ([6, Lemma 2.1]). *Set*

$$\psi(t) = \int_a^t (\alpha(s)\,|u_0(s)| + \beta(s)) \cdot \exp\left(\int_s^t \alpha(l)dl\right) ds.$$

Fix a function $w : [a, +\infty) \to \overline{D(A)}$ *and let* $f(\cdot)$ *be a measurable selection from* $F(\cdot, w(\cdot))$. *The following holds*:

$$\textit{if } |w(t) - u_0(t)| \leqq \psi(t), \quad \textit{then also } |u_f(t) - u_0(t)| \leqq \psi(t).$$

We now need the following

DEFINITION. A subset H of $L^1(I)$ is called *decomposable* if, whenever $u, v \in H$ and E is a measurable set in I, we have $u\chi_E + v\chi_{I\setminus E} \in H$. By Dec $L^1(I)$ we denote the set of all closed, nonempty and decomposable subsets of $L^1(I)$.

The following proposition will play a central role in the proof of our result.

PROPOSITION 3.3 (Fryszkowski). *Let S be a compact metric space and $G : S \to$ Dec $L^1(I)$ be a lower semicontinuous multivalued map with decomposable values. Then there exists $g : S \to L^1(I)$, a continuous selection from G.*

For the proof, see [7] and [4, Thm 3].

In the following $h^*(A, B)$ will denote the separation of a set A from a set B, i.e.

$$h^*(A, B) := \sup_{\eta \in A} d(\eta, B).$$

4. Proof of the main result

In order to apply the selection theorem of Fryszkowski, we need the following result.

PROPOSITION 4.1. *Let F be as in Section 2, I be a compact interval in $[a, +\infty)$ and let K be a compact subset of $L^1(I)$, bounded in $L^\infty(I)$. Then the operator*

$$G : K \to \mathrm{Dec}\, L^1(I)$$

$$x \mapsto \{u \in L^1(I) : u(t) \in F(t, x(t)) \textit{ for a.e. } t \in I\}$$

is well-defined and lower semicontinuous.

Proof. It is easily seen that $G(x_1) = G(x_2)$ whenever $x_1(\cdot) = x_2(\cdot)$ a.e. Moreover $G(x)$ clearly is decomposable, for any $x \in K$. In order to prove the lower semicontinuity, let C be a closed subset of $L^1(I)$ and let (x_n) be a sequence in K converging in $L^1(I)$ to some x_0 in K and such that $G(x_n) \subset C$. We just need to prove that $G(x_0) \subset C$ or, since C is closed, that

(3) $$h^*(G(x_0), G(x_n)) \to 0 \qquad \text{as } n \to \infty.$$

Let $\hat{x}_n$ (resp. $\hat{x}_0$) be Borel functions such that $\hat{x}_n = x_n$ a.e. (resp. $\hat{x}_0 = x_0$ a.e.). We begin by proving the following

Claim.

$$\int_I h^*(F(t, \hat{x}_0(t)), F(t, \hat{x}_n(t)))dt \to 0 \qquad \textit{as } n \to \infty.$$

Proof of the Claim. Set

$$h_n(t) = h^*(F(t, \hat{x}_0(t)), F(t, \hat{x}_n(t))), \qquad \eta_n = \int_I h_n(t)dt.$$

First of all we remark that the maps

$$t \mapsto F(t, \hat{x}_0(t)), \qquad t \mapsto F(t, \hat{x}_n(t))$$

are measurable. Next we show that $h_n(\cdot)$ is measurable. By Theorem 3.5(e) in [8], the map

$$(t, z) \mapsto d(z, F(t, \hat{x}_n(t)))$$

is Carathéodory, and by Theorem 6.5 in the same paper the multivalued map given by

$$\Phi(t) = \{d(z, F(t, \hat{x}_n(t)) : z \in F(t, \hat{x}_0(t))\}$$

is weakly measurable. Hence Theorem 6.6 again in [8] gives the measurability of $h_n(\cdot)$.

Now we will prove that every subsequence (η_{n_k}) of (η_n) has a subsequence converging to 0. In fact $(x_{n_k}(\cdot))$ contains a subsequence (still denoted $(x_{n_k}(\cdot))$) converging to $x_0(\cdot)$ a.e. Then the lower semicontinuity of $F(t, \cdot)$ together with the fact that the values of F are compact implies that $h_{n_k}(t) \to 0$ for a.e. $t \in I$. Moreover, by (F3) and recalling that K is bounded in $L^\infty(I)$,

$$\begin{aligned} h_{n_k}(t) &= h^*(F(t, \hat{x}_0(t)), F(t, \hat{x}_{n_k}(t))) \\ &\leqq h^*(F(t, \hat{x}_0(t)), \{0\}) + h^*(\{0\}, F(t, \hat{x}_{n_k}(t))) \\ &\leqq |F(t, \hat{x}_0(t))| + |F(t, \hat{x}_{n_k}(t))| \\ &\leqq \alpha(t)\{|\hat{x}_0(t)| + |\hat{x}_{n_k}(t)|\} + 2\beta(t) \\ &\leqq 2\{M\alpha(t) + \beta(t)\} \end{aligned}$$

for a suitable constant M. The Lebesgue dominated convergence theorem gives

$$\eta_{n_k} = \int_I h_{n_k}(t)dt \to 0 \qquad \text{as } k \to \infty$$

and this proves the claim.

Finally, in order to prove (3), fix $\bar{u} \in G(x_0)$ and consider the multivalued function

$$\Gamma_n : t \mapsto \bar{B}(\bar{u}(t), h_n(t) + 1/n) \cap F(t, \hat{x}_n(t)),$$

which clearly has closed nonempty values. To show that Γ_n is measurable, it is enough to prove the measurability of the map

$$\Psi : t \mapsto \bar{B}(\bar{u}(t), h_n(t) + 1/n)$$

(see [8, Thm. 4.1]). But Ψ is the composition of the measurable map $t \mapsto (\bar{u}(t), h_n(t) + 1/n)$ with the continuous map $(x, r) \mapsto \bar{B}(x, r)$, hence is measurable. Therefore we can choose a L^1 selection u_n from Γ_n. Clearly $u_n \in G(x_n)$ and we have

$$|u_n - \bar{u}|_{1,I} \leqq \int_I (h_n(t) + 1/n)dt.$$

By the above claim, the right-hand side of this inequality converges to 0 as $n \to \infty$, uniformly in $\bar{u}$. Hence (3) is proved.

Proof of Theorem 2.1. We will follow essentially the proof of Theorem 2.2 in [6]. Define K as the closure in $L^1_{loc}([a, +\infty))$ of the set of those absolutely continuous functions v having the following properties:

(i) $v(a) = x^0$ and $v(t) \in \overline{D(A)}$ $(t \geqq a)$;

(ii) $|v(t) - u_0(t)| \leqq \psi(t)$ $(t \geqq a)$;
(iii) for every interval $I = [\tau, \tau + T]$ $(\tau \geqq a)$,

$$|\dot{v}|_{1,I} \leqq C[(1 + T + N(I))(1 + M(I)) + r^2(\tau)],$$

where

$$M(I) = \exp\left(\int_I \alpha(t)dt\right) \cdot \int_I (\alpha(t)|u_0(t)| + \beta(t))dt + |u_0|_{\infty,I},$$

$$N(I) = M(I) \int_I \alpha(t)dt + \int_I \beta(t)dt,$$

$$r(\tau) = |u_0(\tau)| + 2\int_a^\tau (\alpha(t)|u_0(t)| + \beta(t))\exp\left(2\int_t^\tau \alpha(s)ds\right)dt.$$

It is easily seen (as in [6]) that K is nonempty, convex, compact in $L^1_{loc}([a, +\infty))$ and bounded in $L^\infty_{loc}([a, +\infty))$.

Set, for $n = 1, 2, \ldots,$

$$I_n := [a, a + n], \qquad K_n := \{v|_{I_n} : v \in K\} \subset L^1(I_n).$$

We will construct recursively a sequence of continuous maps $g_n : K_n \to L^1(I_n)$ verifying, for each $x \in K_n$,

$$g_n(x)(t) \in F(t, x(t)) \quad \text{for a.e. } t \in I_n, \tag{4}$$

$$\text{if } n > 1, \quad g_n(x)(t) = g_{n-1}(x)(t) \quad \text{for a.e. } t \in I_{n-1}. \tag{5}$$

Define the operator G_1 in the same way as G with K_1 in place of K, and, for $n > 1$, assuming that g_{n-1} has already been defined, define the operator

$$G_n : K_n \to \operatorname{Dec} L^1(I_n)$$
$$x \mapsto \{u \in L^1(I_n) : u(t) \in F(t, x(t)) \text{ for a.e. } t \in I_n \text{ and}$$
$$u(t) = g_{n-1}(x)(t) \text{ for a.e. } t \in I_{n-1}\}.$$

By Proposition 4.1 and Proposition 3.3, the operator G_1 has a continuous selection g_1. Therefore we can consider the operator G_2, and it is not difficult to see, in view of Proposition 4.1, that it is lower semicontinuous. Applying again Proposition 3.3, we see that G_2 admits a continuous selection g_2 which, by construction, satisfies (4) and (5). Similarly, for any $n > 2$, we obtain g_n from g_{n-1} satisfying (4) and (5). Now we define

$$g : K \to L^1_{loc}([a, +\infty))$$

by setting $g(x)|_{I_n} = g_n(x)$, $n = 1, 2, \ldots$.

Using (4) and (5), it is easy to see that g is well-defined and continuous and satisfies

$$g(x)(t) \in F(t, x(t)) \qquad \text{for a.e. } t \geqq a.$$

To conclude the proof we define, as in [6],

$$s : K \to L^1_{\mathrm{loc}}([a, +\infty)),$$
$$x \mapsto i(g(x)).$$

The map s is continuous and, by Lemma 3.2, $s(K) \subset K$. Since K is compact and convex, the theorem of Schauder–Tichonov yields a fixed point of s, which is a solution to (P).

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References

1. H. Antosiewicz and A. Cellina, *Continuous selections and differential relations*, J. Diff. Eq. **19** (1975), 386–398.
2. H. Attouch and A. Damlamian, *On multivalued evolution equations in Hilbert spaces*, Isr. J. Math. **12** (1972), 373–390.
3. A. Bressan, *On differential relations with lower semicontinuous right-hand side*, J. Diff. Eq. **37** (1980), 89–97.
4. A. Bressan and G. Colombo, *Extensions and selections of maps with decomposable values*, to appear in Studia Math. **90**.
5. H Brézis, *Opérateurs maximaux monotones et semigroupes nonlinéaires*, North-Holland, Amsterdam, 1971.
6. A. Cellina and M. V. Marchi, *Non-convex perturbations of maximal monotone differential inclusions*, Isr. J. Math. **46** (1983), 1–11.
7. A. Fryszkowski, *Continuous selections for a class of non-convex multivalued maps*, Studia Math. **76** (1983), 163–174.
8. C. J. Himmelberg, *Measurable relations*, Fund. Math. **LXXXVII** (1975), 53–72.
9. S. Łojasiewicz jr., *The existence of solutions for lower semicontinuous orientor fields*, Bull. Acad. Polon. Sci. **28** (1980), 483–487.